BIBLIOGRAPHY

1. Sretenskii, L. N., Waves of finite amplitude arising due to periodically distributed pressure. Izv. Akad. Nauk SSSR, OTN №4, 1953.
2. Sekerzh-Zen'kvoich, Ia. I., On the theory of steady waves of finite amplitude produced by pressure distributed periodically over the surface of an infinitely deep stream of heavy fluid. Dokl. Akad. Nauk SSSR, Vol. 180, №2, 1968.
3. Sekerzh-Zen'kovich, Ia. I., Steady waves of finite amplitude due to pressure distributed periodically over the surface of a stream of heavy fluid of finite depth. Dokl. Akad. Nauk SSSR, Vol. 180, №3, 1968.
4. Nekrasov, A. I., The Exact Theory of Steady Waves on the Surface of a Heavy Fluid, pp. 1-96, Moscow, Izd. Akad. Nauk SSSR, 1951.
5. Vainberg, M. M. and Trenogin, V. A., The Liapunov and Schmidt methods in the theory of nonlinear equations and their further development. Uspekhi Mat. Nauk Vol. 17, №2, 1962.
6. Goursat, E., A Course in Mathematical Analysis. Vol. 3, Part 2, p. 121. Moscow-Leningrad, Gostekhteoretizdat, 1934.

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SMALL TORSIONAL OSCILLATIONS OF AN ELASTICALLY CONSTRAINED RIGID CIRCULAR CYLINDER FILLED WITH A VISCOUS FLUID

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A. B. IVANOV
(Leningrad)
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This paper considers the rotatory motion of a cylinder of finite height, filled with a viscous incompressible fluid, and subjected to an elastic moment, the cylinder being initially at rest in a position obtained from the equilibrium position by rotation through a small angle . The solution of the problem is constructed in the form of a Laplace-Mellin integral. The vibration spectrum of the system is studied and a spectral expansion for the solution obtained, the latter yielding a description of the nature of the cylinder's motion for various values of the parameters involved.

This problem was solved earlier under the assumption that the oscillations decay harmonically, which assumption is valid in a certain time interval when the ratio of the moments caused by the viscous friction forces to the maximal elastic moment is sufficiently small [1 and 2]. A general investigation of the characteristic equation for the oscillations was not carried out, and the problem in the large (with account taken of the initial conditions) was not posed. The present paper fills this gap.

It is established that for any positive values of the parameters the rigid cylinder passes through the equilibrium position. Depending on the values of the parameters, two things can happen: (1) the cylinder passes through the equilibrium position an infinite number of times, or (2) the cylinder passes through the equilibrium position an odd number of times and then approaches the equilibrium position as time approaches infinity, from the side opposite that of the initial position.

The investigation of analogous problems for a sphere or an infinite cylinder filled with a viscous fluid is significantly simpler since, unlike the problem for a cylinder of finite height, the fluid motion will depend on one spatial coordinate only [3]. The quanlitative results of such an investigation do not differ from those obtained for the finite cylinder. Apparently, small torsional oscillations of an arbitrary elastically constrained rigid surface of revolution containing viscous fluid or immersed in such a fluid follow the essential pattern of such oscillations in the simplest concrete problems, where the fluid moves as a family of quasirigid surfaces.

1. Formulation of the problem. Integral representation and differential properties of the solution. A rigid right circular cylindrical surface of radius R_*, height $2H_*$ and moment of inertia K_*, filled with a homogeneous incompressible fluid of viscosity η_* and density μ_* is axisymmetrically attached to an elastic filament of torsional rigidity $M_* = K_* k_{0*}^2$ (k_{0*} is the frequency of the free undamped harmonic oscillations of the cylinder without fluid). At the initial moment of time $t_* = 0$ the system is at rest and the cylinder is twisted at a small angle A_0 relative to the equilibrium position. The parameters A_0, R_*, H_*, K_*, M_*, η_*, μ_* are positive. The basic quantity under study is the angular velocity $\omega_{0*}(t_*)$ of the solid cylinder for $t_* > 0$.

We pass to dimensionless quantities

$$t = k_{0*}t_*,\quad H = \frac{H_*}{R_*},\quad \omega_0 = \frac{\omega_{0*}}{k_{0*}},\quad \eta = \frac{2\pi R_*^3}{K_* k_{0*}}\eta_*,\quad \mu = \frac{2\pi R_*^5}{K_*}\mu_*,\quad \nu = \frac{\eta}{\mu}$$

On a meridianal half-plane we introduce dimensionless rectangular coordinates r, y such that the portion inside the cylindrical surface considered is defined by the inequalities $0 < r < 1,\ -H < y < H$. The angular velocity of the fluid about the axis $r = 0$, relative to k_{0*}, is denoted by $\omega(t, r, y)$.

As solution of the problem, we seek a function $\omega(t, r, y)$ defined in $D\,(t \geqslant 0,\ 0 < r \leqslant 1,\ |y| \leqslant H)$ and satisfying the following conditions:

1) the function $\omega(t, r, y)$ is continuous in the region D and vanishes for $t = 0$, $0 < r \leqslant 1$, $|y| \leqslant H$;

2) for $t > 0$ and in D there exist continuous derivatives ω_t, ω_r, ω_y; inside D, ω also has continuous derivatives ω_{rr}, ω_{yy}, and satisfies the equation

$$\frac{1}{\nu}\omega_t = \omega_{rr} + \frac{3}{r}\omega_r + \omega_{yy} \tag{1.1}$$

3) in each region $(\varepsilon < t < t_0,\ 0 < r < 1,\ |y| < H)$, where ε and t_0 are arbitrary positive numbers, the following estimates are valid:

$$|\omega| < C,\qquad |\omega_r| < C_1 r^{-1},\qquad |\omega_y| < C_1 r^{-1},$$
$$|\omega_t| < C_1 r^{-3}\,|\omega_{yy}| < C_2 r^{-3}(H - |y|)^{-1/2}$$

where C depend only on t_0, and C_1, C_2 only on ε and t_0;

4) the function $\omega(t, 1, y) = \omega(t, r, -H) = \omega(t, r, H) \equiv \omega_0(t)$ for $t > 0$ satisfies the equation

$$\frac{d\omega_0}{dt} + A_0 + \int_0^t \omega_0(\tau)\,d\tau + \eta\Big(\int_{-H}^{H} \omega_r|_{r=1}\,dy - \int_0^1 r^3\omega_y|_{y=-H}\,dr + \int_0^1 r^3\omega_y|_{y=H}\,dr\Big) = 0 \tag{1.2}$$

The concept of a solution is defined here in such a way that its uniqueness may be established by examining the energy integral. Such a definition is more natural physically than that based on a constructive procedure involving the Laplace transform. The latter leads to the following result:

$$\omega(t, r, y) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{-A_0 \Psi(r, y, z)\, e^{zt}\, dz}{\varphi(z)} \tag{1.3}$$

$$\gamma > \max \operatorname{Re} \lambda, \quad \varphi(\lambda) = 0$$

$$\Psi(r, y, z) = \frac{1}{r}\left[\frac{I_1(br)}{I_1(b)} - \sum_{n=1}^{\infty} \frac{2}{\alpha_n J_0(\alpha_n)} \frac{b^2 \operatorname{ch}(b_n y)}{b_n^2 \operatorname{ch}(b_n H)} J_1(\alpha_n r)\right] =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{2}{J_0(\alpha_n)} \frac{b^2}{b_n^2}\left[1 - \frac{\operatorname{ch}(b_n y)}{\operatorname{ch}(b_n H)}\right] \frac{J_1(\alpha_n r)}{\alpha_n r}$$

$$\varphi(z) = z^2 + 1 + 2H\eta z\left[-1 + b\frac{I_1'(b)}{I_1(b)} + 2b^4 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2 b_n^2} \frac{\operatorname{th}(b_n H)}{b_n H}\right] =$$

$$= \left(1 + \frac{H\mu}{2}\right) z^2 + 1 + \frac{4H\mu}{\nu} z^3 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2 b_n^2}\left[\frac{\operatorname{th}(b_n H)}{b_n H} - 1\right]$$

$$b_n = \sqrt{b^2 + \alpha_n^2}, \quad b = \sqrt{z/\nu}, \quad 0 < \alpha_1 < \alpha_2 < \ldots, \quad I_1(u) = -iJ_1(ui).$$

Here $J_0(u)$ and $J_1(u)$ are Bessel functions and α_n the positive roots of $J_1(u)$. From the asymptotic representation of Bessel functions follow the following facts, essential for the investigation of $\Psi(r, y, z)$ and $\varphi(z)$: the quantities

$$\frac{\alpha_n}{n}, \qquad \sqrt{n}\,|J_0(\alpha_n)|, \qquad \frac{\alpha_n - \alpha_m}{n - m}, \qquad n \neq m$$

are bounded between two positive numbers, independently of n and m.

The second expression for $\varphi(z)$ is obtained from the first as a consequence of the equations

$$g(u) \equiv \frac{1}{2u}\left[-1 + \sqrt{u}\frac{I_1'(\sqrt{u})}{I_1(\sqrt{u})}\right] = \sum_{n=1}^{\infty} \frac{1}{u + \alpha_n^2} = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} - \sum_{n=1}^{\infty} \frac{u}{\alpha_n^2 (u + \alpha_n^2)}$$

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} = g(0) = \frac{1}{8}$$

Clearly the poles of $\varphi(z)$ are simple, negative, and are given by the double sequence

$$\zeta_{nm} = -\nu\alpha_n^2 - (2m - 1)^2 \frac{\pi^2 \nu}{4H^2} \qquad (n = 1, 2, \ldots;\ m = 1, 2, \ldots)$$

We enumerate the poles in the order of their decrease, denoting coincident values of ζ_{nm} (if such exist) by the same number

$$0 > -\nu\alpha_1^2 > \zeta_1 > \zeta_2 > \cdots > \zeta_p > \zeta_{p+1} > \cdots$$

For $0 \leqslant r < 1$, $|y| < H$ the function $\Psi(r, y, z)$ is meromorphic in z. Its poles are simple and all are contained in the set of poles ζ_p of $\varphi(z)$. The limit conditions for Ψ when $z \neq \zeta_p$, $p = 1, 2, \ldots$ take the form

$$\Psi(1, y, z) \equiv \Psi(r, -H, z) \equiv \Psi(r, H, z) \equiv 1$$

$$|y| \leqslant H, \qquad 0 \leqslant r \leqslant 1$$

In the sector $S\,(-\pi+\varepsilon \leqslant \arg z \leqslant \pi-\varepsilon)$, where ε is a fixed arbitrarily small positive number, the following asymptotic relations as $z \to \infty$ are easily obtained by estimating the series by integrals over n :

$$\varphi(z) = z^2 + O(z^{3/2}), \qquad \Psi(r, y, z) = O(z^{3/4}), \qquad \Psi_r = \frac{1}{r} O(z^{3/4}), \qquad \Psi_y = \frac{1}{r} O(z^{3/4})$$

The three latter relations hold uniformly in the rectangle $(0 < r \leqslant 1,\ |y| \leqslant H)$; Ψ is continuous in $(z \in S,\ 0 \leqslant r \leqslant 1,\ |y| \leqslant H)$, and Ψ_r and Ψ_y in $(z \in S,\ 0 < r \leqslant 1,\ |y| \leqslant H)$. Consequently : (1) the function ω is continuous in $(t \geqslant 0,\ 0 \leqslant r \leqslant 1,\ |y| \leqslant H)$, (2) the functions ω_r and ω_y are continuous at least in $(t \geqslant 0, 0 < r \leqslant 1,\ |y| \leqslant H)$, in which domain $|\omega_r| < C_1 r^{-1}$, $|\omega_y| < C_1 r^{-1}$, (3) the functions $\omega, \omega_r, \omega_y$ vanish in the entire cross section $(t = 0,\ 0 \leqslant r \leqslant 1,\ |y| \leqslant H)$. It is easy to obtain the estimate

$$|\omega_{yy}| < C_2\,[r^{-3/2} \ln^2 (H - |y|) + 1], \quad t \geqslant 0,\ 0 < r \leqslant 1,\ |y| < H$$

The quantities C_1 and C_2 here turn out to be finite and constant, if the time interval $(0, t_0)$, $t_0 < \infty$, is fixed (since ω dies out in time, one may actually choose C_1 and C_2 to be independent of the time interval).

For $t \geqslant 0,\ 0 \leqslant r < 1,\ |y| < H$ the function ω is infinitely differentiable with respect to all variables (and all derivatives vanish at $t = 0$) ; the differentiation may be taken under the integral sign in (1.3); and it may be verified immediately that the function given by (1.3) satisfies (1.1).

To investigate the differential properties of ω for $t > 0$ it is convenient to deform the contour of integration in (1.3), keeping all roots of $\varphi(z)$ to the left of it, into a new contour with the property that for sufficiently large $|z|$ its points all lie within the sectors $(\pi - \varepsilon > \arg z > \frac{1}{2}\pi + \varepsilon)$, $(-\pi + \varepsilon < \arg z < -\frac{1}{2}\pi - \varepsilon)$, $\varepsilon > 0$. Thus one sees that ω is analytic in t for $t > 0,\ 0 \leqslant r \leqslant 1,\ |y| \leqslant H$, and in the remaining variables for $t > 0,\ 0 \leqslant r \leqslant 1,\ |y| < H$. In particular the angular velocity $\omega_0(t)$ of the cylinder will be analytic for $t > 0$. All derivatives with respect to t, $\omega_t^{\,n}$, $n = 1, 2, \ldots$, $t > 0,\ 0 \leqslant r \leqslant 1,\ y \leqslant H$ may be found (after making the indicated contour change) by differentiating under the integral sign (in computing ω_t it is not necessary to change the integration contour). The verification of Eq. (1.2) for $t > 0$ now proceeds without difficulty, thus completing the proof that the integral (1.3) is the solution of the problem.

According to (1.2) the angular acceleration $\omega_0'(t)$ is continuous for $t \geqslant 0$ and reduces to $-A_0$ at $t = 0$, so that ω_t undergoes a jump from the value 0 to the value $-A_0$ as the boundary (corresponding to the rigid cylinder) is approached from the inside at $t = 0$.

2. The spectrum of the problem. We pass to an investigation of the characteristic equation $\varphi(z) = 0$ of the cylinder's oscillations for arbitrary positive values of the parameters H, η, ν. The set of roots of $\varphi(z)$ exhausts the points of the spectrum i.e. the values of the complex parameter z for which Eq. (1.1) has bounded solutions of the form $\omega = \mathrm{Re}\,[e^{zt} f\cdot(r, y, z)]$ with limit condition of the form (1.2).

The fundamental result of the present article is formulated as follows.

Theorem. For any positive values of the parameters η, H, ν the set of roots of $\varphi(z)$ consist of a countable number of negative numbers

$$k_p \in (\zeta_{p+1}, \zeta_p), \qquad \varphi'(k_p) < 0 \qquad (p = 1, 2, \ldots)$$

and pairs of complex conjugate numbers

$$k_0 = -\alpha + \beta i, \ k_0 = -\alpha - \beta i, \ \alpha > 0, \ \beta > 0, \ \varphi'(k_0) \neq 0$$

If the parameters $H > 0$, $\nu > 0$ are fixed and the parameter η, $0 \leqslant \eta < \infty$

increases, then the root $k_0(\eta)$ proceeds from the initial point $z = i$, and traverses a continuous trajectory in the plane without multiple points, which is asymptotically tangent to the imaginary axis at the limiting point $z = 0$ as $\eta \to \infty$.

The proof of this assertion is based on three lemmas.

Lemma 2.1. On the z plane there exists a sequence of circles Γ_m, $m = 1, 2, \ldots$, with center at $z = 0$ and radii increasing to infinity, such that on $\Gamma_1 + \Gamma_2 + \Gamma_3 + \ldots$ as $z \to \infty$

$$\varphi(z) = z^2 + o(z^2)$$

Lemma 2.2. In each interval (ζ_{p+1}, ζ_p) between the poles ζ_p $(p = 1, 2, \ldots)$ the function $\varphi(z)$ has one and only one root k_p. On the ray $\zeta_1 < z < \infty$ the indicated function is positive.

Lemma 2.3. For any fixed $H > 0$, $\nu > 0$ and sufficiently large $\eta \to \infty$, in the circle $|z| \leqslant \varepsilon(\eta)$ $(\lim \varepsilon(\eta) = 0, \lim \eta\, [\varepsilon(\eta)]^2 = \infty)$ the function $\varphi(z)$ has exactly two roots k_0, $\bar{k}_0$, where

$$k_0 = \sqrt{2\nu/\eta H}\, i\, [1 + o(1)]$$

The principal difficulty consists in the proof of 2.1 and 2.2. The theorem itself is proved very simply from the lemmas. Using Lemma 2.1 with account taken of the argument principle, one may choose a contour number m_0 such that for all $m \geqslant m_0$ the difference between the number of roots and poles of $\varphi(z)$ inside Γ_m is equal to two; hence by Lemma 2.2 the function $\varphi(z)$ has two imaginary roots $k_0, \bar{k}_0$ besides the negative roots, and has no others. The asymptotic behavior of the root k_0 for fixed positive H, ν and $\eta \to \infty$ is described in Lemma 2.3. The continuity of the trajectory $k_0(\eta)$, $\eta \in [0, \infty]$, $k_0(0) = i$, $k_0(\infty) = 0$, is obvious. The absence of multiple points on it follows from the possibility of expressing η uniquely in terms of k_0.

The condition $\operatorname{Re} k_0 < 0$, physically evident, may be formally justified, for example, as follows. In (1.3) we separate out the sum of those residues of the integrand which corresponds to k_0, $\bar{k}_0$:

$$\omega(t, r, y) = B_0 e^{-\alpha t} \cos(\beta t + \vartheta) + \chi(t, r, y)$$

$$\vartheta = \arg \frac{\Psi(r, y, k_0)}{\varphi'(k_0)} \qquad B_0 = \frac{-2A_0 |\Psi(r, y, k_0)|}{|\varphi'(k_0)|}$$

$$\chi(t, r, y) = \frac{1}{2\pi i} \int_Q \frac{-A_0 \Psi}{\varphi} e^{zt}\, dz$$

The contour Q is formed from two rays

$$\arg z = \pi + \varepsilon,\ \infty > |z| \geqslant 0; \qquad \arg z = \pi - \varepsilon,\ 0 \leqslant |z| < \infty$$
$$(0 < \varepsilon < \pi/2,\ \varepsilon < \pi - \arg k_0,\ 0 < \arg k_0 < \pi)$$

The functions $r\chi_r$ and $r\chi_y$ approach zero as $t \to \infty$ uniformly in r, y, and the function $\Psi(r, y, k_0)$ depends on the variables r, y since it satisfies the equation

$$\Psi_{rr} + \frac{3}{r}\Psi_r + \Psi_{yy} - \frac{k_0}{\nu}\Psi = 0$$

and takes on the value one on the surface of the rigid cylinder. Therefore the condition $\operatorname{Re} k_0 \geqslant 0$ contradicts the following energy equality which follows from the formulation of the problem and which is true for all $t \geqslant 0$:

$$\omega_0^2(t) + 2A_0 \int_0^t \omega_0(\tau)\, d\tau + \left[\int_0^t \omega_0(\tau)\, d\tau\right]^2 +$$
$$+ \frac{\eta}{\nu} \int_{-H}^{H} \int_0^1 r^3 \omega^2\, dr\, dy + 2\eta \int_0^t \int_{-H}^{H} \int_0^1 r^3 [(\omega_r)^2 + (\omega_y)^2]\, dr\, dy\, dt = 0$$

Proof of Lemma 2.1. We fix arbitrary positive values of the parameters H, η, ν. We replace the variable z by $u = bi$. In each interval (α_m, α_{m+1}), $m = 1, 2, \ldots$, between positive roots of $J_1(u)$ we select a subinterval $(\beta_m$ such that each of the segments (α_m, β_m), (β_m, β_{m+1}), $(\beta_{m+1}, \alpha_{m+1})$ will have a l gth greater than some positive number l, the same for all m.

For some δ, $0 < \delta < 1/4$, in each half-plane

$$w_n = b_n H \quad (n = 1, 2, \ldots), \; 0 \leqslant \arg b_n \leqslant \pi$$

we construct disks $\{\varepsilon_q^{(m)}\} = \{|w_n - (2q-1)\pi i/2| \leqslant \delta/m\}$ $(q = 1, 2, \ldots)$

It is easily verified that the image of the disk $\{\rho_q^{(m)}\}$ under the transformation $w_n \to u$, $|\arg u| \leqslant \pi/2$ lies inside the disk

$$\{R_{nq}^{(m)}\} = \left\{|u - \xi_{nq}| \leqslant K_{nq}\frac{\delta}{m}\right\}$$

$$\varepsilon_{nq} = \left(\alpha_n^2 + \frac{\pi^2}{4H^2}(2q-1)^2\right)^{1/2} \qquad K_{nq} = \frac{\pi(2q-1)}{H^2\xi_{nq}} < \frac{2}{H}$$

and that the sum of diameters of the circles $\{R_{nq}^{(m)}\}$ with centers at $\xi_{nq} \in (\beta_m, \beta'_{m+1})$ satisfies

$$S_m < 4\left(\frac{L}{\pi}\frac{\sqrt{L}}{\sqrt{l}} + \frac{1}{H}\right)\delta, \qquad L = \sup(\alpha_{m+1} - \alpha_m) \qquad (m = 1, 2, \ldots)$$

We finally choose δ such that $S_m < 1/2\, l - 4\delta H^{-1}$. Then for each (β_m, β_{m+1}) the sum $\{\Sigma_m\}$ of its subintervals lying outside the disks $\{R_{nq}^{(m)}\}$, has measure $\operatorname{mes}\{\Sigma_m\} > l/2$, so certainly is not empty.

On circles P_m with center $u = 0$ intersecting (β_m, β_{m+1}) at the points $\gamma_m \in \{\Sigma_m\}$, $m = 1, 2, \ldots$, the following estimates hold uniformly in m, γ_m, N

$$\left|\frac{J_1'(u)}{J_1(u)}\right| < C_3, \qquad |\operatorname{th} w_n| < C_4|u|$$

$$|\varphi(z) - z^2| < 1 + C_5(|u|^2 + |u|^3 + |f(u)|), \qquad f(u) \equiv \sum_{n=1}^{\infty}\frac{u^6 \operatorname{th} w_n}{\alpha_n^2(u^2 - \alpha_n^2)^{3/2}}$$

For $m > 2N + 1$, the following holds on P_m uniformly in m, γ_m and N:

$$|f(u)| < C_6\Bigg\{|u|^3\sum_{n=1}^{n=N}\frac{|\operatorname{th} w_n|}{n^2} + |u|^4\sum_{n=N+1}^{n=[m/2]}\frac{1}{n^2} + $$
$$+ |u|^{7/2}\left[\sum_{n=[m/2]+1}^{n=m}\frac{1}{(1+m-n)^{3/2}} + \sum_{n=m+1}^{\infty}\frac{1}{(n-m)^{3/2}}\right]\Bigg\}$$

We suppose given any sequence of positive numbers $\varepsilon_s \to 0$, $s = 1, 2, \ldots$ for each ε_s we define $N = N_s$ so that

$$\sum_{n=N_s+1}^{\infty}\frac{1}{n^2} < \varepsilon^s$$

For $n \leqslant N_s$, $s = 1, 2, \ldots$, we construct disks

$$\{T_{nq}^{(s)}\} = \{|u - \xi_{nq}| \leqslant K_{nq}\delta_s\}$$

where $\delta_s \leqslant \delta$ are positive numbers determined so that the set $\{\Sigma_m^{(1)}\}$, which remains after removing from $\{\Sigma_m\}$ those points belonging to the disks $\{T_{nq}^{(s)}\}$, is nonvoid for every $m = 1, 2, \ldots$. Finally we choose the circles P_m so that $\gamma_m \in \{\Sigma_m^{(1)}\}$. Then

$$|f(u)| < C(s)|u|^3 + C_6\varepsilon_s|u|^4 + C_7|u|^{7/2}$$

$$u \in P_m, \qquad s = 1, 2, \ldots, \qquad m = 2N_s + 2, 2N_s + 3, \ldots$$

On the contours Γ_m, images in the z plane of the circles P_m, the following holds for any fixed $s = 1, 2, \ldots$ as $z \to \infty$:

$$|\varphi(z) - z^2| = C_8\varepsilon_s|z|^2 + o(|z|^2)$$

which is equivalent to the assertion of Lemma 2.1, since C_8 does not depend on s.

Proof of Lemma 2.2. For $z < 0$

$$\varphi(z) = 1 + \frac{4\mu\nu^2}{H^2}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\varphi_n(z), \qquad \varphi_n(z) = \Phi_n(x_n)$$

$$\Phi_n = (x_n^2 + c_n^2)^3\frac{\operatorname{tg} x_n - x_n}{x_n^3} + \alpha_n^2 r_n (x_n^2 + c_n^2)^2$$

$$z < -\nu\alpha_n^2, \qquad x_n = b_n Hi < 0$$

$$\Phi_n = (c_n^2 - x_n^2)^3\frac{x_n - \operatorname{th} x_n}{x_n^3} + \alpha_n^2 r_n (c_n^2 - x_n^2)^2$$

$$-\nu\alpha_n^2 \leqslant z < 0, \qquad 0 \leqslant x_n = b_n H < c_n \doteq \alpha_n H$$

Here r_n are chosen so that

$$\sum_{n=1}^{\infty} r_n = \frac{1}{4H\mu} + \sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}, \qquad r_n \geqslant \frac{1}{\alpha_n^2}$$

Here for $x < 0$ and $\cos x \neq 0$ (by convention we omit the index n)

$$\Phi'(x)\, x^4\cos^2 x \leqslant (x^2 + c^2)\,[x^5 + 3x^4\sin x\cos x +$$
$$+ 2c^2x^3\sin^2 x + c^4(1 + 2\cos^2 x)\,x^3 - 3c^4\sin x\cos x]$$

If $x \leqslant -\pi/2 < -3/2$

$$x^5 + 3x^4\sin x\cos x < 0, \qquad f(x) \equiv (1 + 2\cos^2 x)\,x - 3\sin x\cos x < 0$$

For $-\pi/2 < x < 0$ the function $f(x)$ is also negative, since it increases monotonically from $-\pi/2$ to zero

$$f'(x) = 2\sin 2x(\operatorname{tg} x - x) > 0$$

Therefore for $x_n < 0$, $\cos x_n \neq 0$, $n = 1, 2, \ldots$

$$\Phi_n'(x_n) < 0, \qquad \varphi_n'(z) = -\frac{H^2}{2\nu x_n}\Phi_n'(x_n) < 0$$

Using the equation

$$\left(\frac{x - \operatorname{th} x}{x^3}\right) = x^{-3}\operatorname{ch}^{-2}x\sum_{k=2}^{\infty} a_k x^{2k}, \qquad a_k > 0, \qquad k = 2, 3, 4, \ldots$$

we also obtain that for $0 \leqslant x_n < c_n$

$$\varphi_n'(z) = \frac{H^2}{2\nu x_n}\Phi_n'(x_n) < 0$$

Thus, for $z < 0$, $z \neq \zeta_p$, $p = 1, 2, \ldots$, each of the functions $\varphi_n(z)$ has a negative derivative and $\varphi'(z) < 0$. On the interval $(\zeta_1, 0)$ the function $\varphi(z)$ decreases monotonically from $+\infty$ to 1 and, consequently, is positive.

If $z > 0$, then $\varphi(z) > 0$, for example, because

$$-1 + b\frac{I_1'(b)}{I_1(b)} = 2\sum_{n=1}^{\infty}\frac{z}{z + \nu\alpha_n^2} > 0$$

Proof for Lemma 2.3. For $H = \text{const} > 0$, $\nu = \text{const} > 0$, $\eta \to \infty$, $z \to 0$

$$\varphi(z) = c\eta z^2 + 1 + o(c\eta z^2), \qquad c = \frac{H}{2\nu}$$

On the circle $|z| = \varepsilon(\eta)$ the quantity $c\eta z^2 \to \infty$ as $\eta \to \infty$, therefore for sufficiently large η the function $\varphi(z)$ has in the disk $|z| \leqslant \varepsilon(\eta) < \nu\alpha_1^2$ two and only two roots k_0, $\bar{k}_0$, $\operatorname{Im} k_0 > 0$, for which $c\eta k_0^2 [1 + o(1)] = -1$

Note. In dimensional quantities, $c\eta = K_*' / K_*$, where $K_*' = \pi\mu_* R_*^4 H_*$ is the moment of inertia of the fluid mass in the cylinder. The obtained asymptotic relation $k_0 = \sqrt{K_*'/K_*}\, i\, [1 + o(1)]$ has a clear physical sense: for large viscosity η_* the cylinder with the fluid may undergo oscillations as a single solid body with moment of inertia $K_* + K_*'$, and for $K_*' \gg K_*$ the period of the oscillations grows to approximately $\sqrt{K_*' / K_*}$ times the period of the oscillations of the cylinder without the viscous fluid.

3. Spectral expansion of the angular velocity of the cylinder. From Lemma 2.1 it follows that the integral (1.3) for $\Psi = 1$ may be represented for all $t \geqslant 0$ by a series of residues of the integrand at its poles k_0, $\bar{k}_0$, k_p, $p = 1, 2, \ldots$

$$\omega_0(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{-A_0 e^{zt}}{\varphi(z)}\, dz = a_0 e^{-\alpha t} \cos(\beta t + \vartheta_0) + \sum_{p=1}^{\infty} a_p e^{k_p t} \qquad (3.1)$$

$$\vartheta_0 = -\arg \varphi'(k_0), \qquad k_0 = -\alpha + \beta i \quad (\alpha > 0,\ \beta > 0), \qquad k_p < 0$$

$$\sum_{p=1}^{\infty} a_p = -a_0 \cos \vartheta_0, \quad a_p = -\frac{A_0}{\varphi'(k_p)} > 0, \quad a_0 = -\frac{2A_0}{|\varphi'(k_0)|} < 0$$

The angular deflection of the rigid cylinder is found by termwise integration of the series (3.1)

$$\underset{t \geqslant 0}{A(t)} = A_0 + \int_0^t \omega_0(\tau)\, d\tau = c_0 e^{-\alpha t} \cos(\beta t + \vartheta_1) + \sum_{p=1}^{\infty} c_p e^{k_p t} \qquad (3.2)$$

$$c_p = \frac{a_p}{k_p} < 0, \quad c_0 \cos \vartheta_1 = A_0 - \sum_{p=1}^{\infty} c_p, \quad |c_0| > A_0, \quad |c_0| > |c_p| \quad (p = 1, 2, \ldots)$$

We consider the principal terms in (3.2) as $t \to \infty$

$$A(t) \approx c_0 e^{-\alpha t} \cos(\beta t + \vartheta_1) + c_1 e^{k_1 t}$$

If $-\alpha \geqslant k_1$, then the arbitrarily large t, $A(t)$ oscillates between positive and negative values. If, however, $-\alpha < k_1$, then for sufficiently large t, $A(t) < 0$, and therefore, being an analytic function, $A(t)$ changes sign on the entire ray $t > 0$ a finite, and in fact, an odd number of times. The first case is realized, for example, when for fixed H, ν, the parameter η is sufficiently small; and the second, when the values of H, η^2 / ν are fixed and the parameter ν is small; or, for example, when H and $\mu = \eta/\nu$ are fixed and η is small.

BIBLIOGRAPHY

1. Okaya, T. and Hasegawa, M., On the motion of liquid in a hollow cylinder caused by the torsional vibration. Japan. Journ. Phys., Vol. 11, №1, pp. 13-22, 1936.
2. Shvidkovkii, E. G., Certain Questions on the Viscosity of Molten Metals. M., Gostekhizdat, 1955.
3. Ivanov, A. B. and Rusanov, B. V., Nonstationary rotation of a viscous fluid by quasirigid surfaces. Vest. Leningr. Univ; №13, issue 3, pp. 82-87, 1968.

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